Lecture 16

Plan:

1) Matroid opt.
   (see lec 15 notes)

2) Matroid polytopes

3) Next time? Matroid intersection

Pset 4 due Mon Apr 26
Pset 5 probs due May 13

More preliminaries:

Rank function

- Analogous to rank of matrices
- rank function $r_M : 2^E \to \mathbb{N}$

of matroid $M = (E, \mathcal{I})$ is

$$\boxed{r_M(X) := \max\{|Y| : Y \subseteq X,\ Y \in \mathcal{I}\}.}$$

= size of largest independent set in X

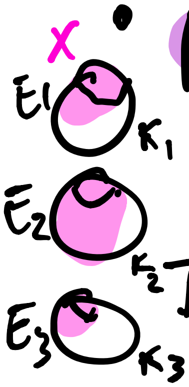
= size of any independent set that is maximal in X. (all max'l indep sets in X have same size, maximum in X $\Rightarrow$ max'l).

- sometimes just $r := r_M$.

Examples

- linear matroid: $r(X) = \operatorname{rank}(A_X)$ ← usual rank.

- partition matroid: Recall



for $E = E_1 \dot\cup \cdots \dot\cup E_\ell$, $k_1 \dots k_\ell$

$$\mathcal{I} = \{X \subseteq E : |X \cap E_i| \le k_i \ \forall i = 1 \dots \ell\}.$$

$$\boxed{r(X) = \sum_{i=1}^{\ell} \min\{|E_i \cap X|, k_i\}}$$

- **Graphic matroid**: M_G, $G=(V,E)$.

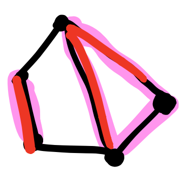
For $F \subseteq E$

$$r(F) = n - K(V,F)$$

$K(V,F) :=$ # connected components of graph w/ vertices V edges F.

e.g.

F



$$r(F) = 5 - 2 = 3$$

Properties of rank function

Let r be rank function of matroid.

(R1) $0 \leq r(X) \leq |X|$

(R2) <u>monotonicity</u>: $X \subseteq Y$
$\Rightarrow r(X) \leq r(Y)$

(R3) <u>submodularity</u>:

$r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$

Ex. try to prove diminishing returns for linear matroid.

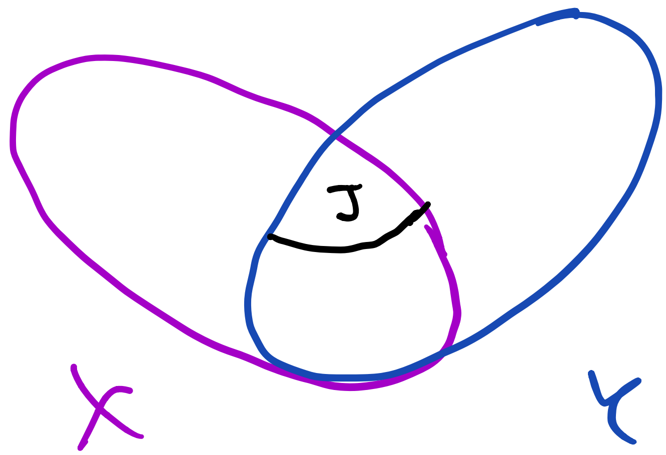
typo before!

<u>**Proof of R3:**</u> • Let $X, Y \subseteq E$.

• We want to show

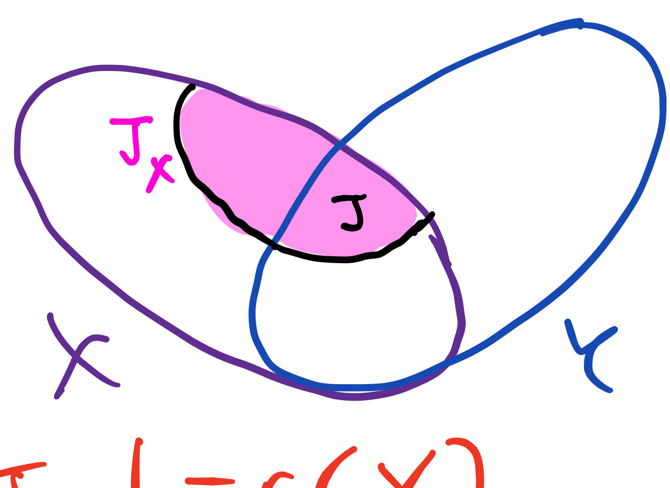
Build chain
$$\begin{array}{ccccc} J & \subseteq & J_X & \subseteq & J_{XY} \\ \text{in} & & \cap\text{i} & & \cap\text{i} \\ X \cap Y & \subseteq & X & \subseteq & X \cup Y \end{array}$$

- Let J max'l indep subset of $X \cap Y$.



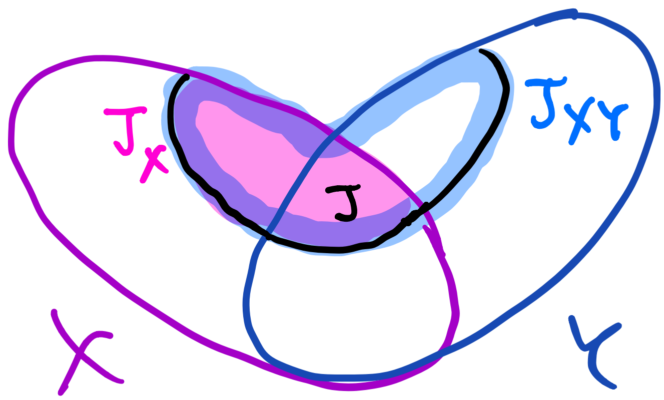
$\Rightarrow |J| = r(X \cap Y)$ (by ☆)

- Extend J to J_X max'l indep subset of X.



$\Rightarrow |J_X| = r(X)$.

- Extend J_X to J_{XY} max'l independent subset of $X \cup Y$



$$\Rightarrow |J_{XY}| = r(X \cup Y).$$

- Note

$$\begin{array}{ccccc} X \cap Y & \subseteq & X & \subseteq & X \cup Y \\ \cup\!| & & \cup\!| & & \cup\!| \\ J & \subseteq & J_X & \subseteq & J_{XY} \\ \| & & \| & & \\ (X\cap Y)\cap J_{XY} & & X \cap J_{XY}. & & \end{array}$$

by J max'l in $X \cap Y$

J_X max'l in X.

☆

i.e.

- <u>submodularity</u> $\Longleftrightarrow$

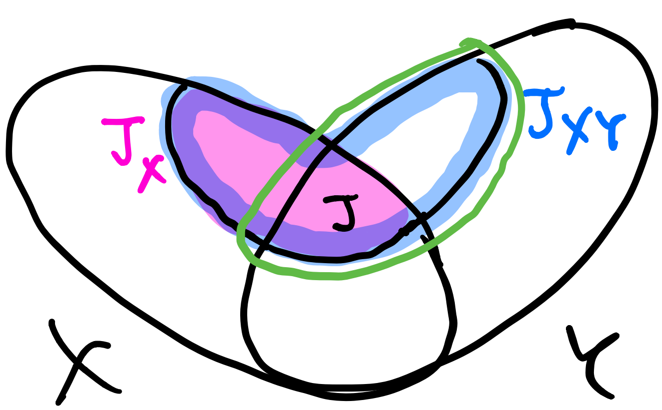
$$r(X)+r(Y)\geq r(X\cap Y)+r(X\cup Y).$$

$$\Longleftrightarrow |J_X|+r(Y)\geq |J|+|J_{XY}|$$

$$\Longleftrightarrow r(Y)\geq |J|+|J_{XY}|-|J_X|.$$

- To prove, find indep set in Y:
  use $J_{XY}\cap Y$ (indep b/c J_{XY} is).

$$\Rightarrow r(Y)\geq |J_{XY}\cap Y|.$$



$J_{XY}\cap Y.$

- **Claim:**

$$\boxed{|J_{XY} \cap Y| = |J_{XY}| + |J| - |J_X|}$$

Pf of Claim: $|J_{XY} \cap Y|$

$$= |(J_{XY} \cap Y) \setminus X| + |(J_{XY} \cap Y) \cap X|$$

$$= |(J_{XY} \setminus X) \cap Y| + |J_{XY} \cap (Y \cap X)|$$

$J_{XY} \subseteq X \cup Y$

$$= |J_{XY} \setminus X| + |J|$$

$$= |J_{XY} \setminus J_X| + |J|$$

$\downarrow J_X \subseteq J_{XY}$.

$$= |J_{XY}| - |J_X| + |J|. \quad \square$$

Comment: pic. for slack is Vámos matroid (shaded parts are circuits). NOT REPRESENTABLE

Span:

- Given $M=(E,I)$, span of $S \subseteq E$ is

$$\boxed{\mathrm{span}(S) := \{e \in E : r(S+e) = r(S)\}}$$

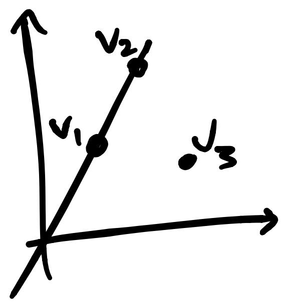
i.e. all elements that do not increase rank of S when added.

e.g. linear matroid, $v_1, \dots, v_m \in \mathbb{F}^n$:

$$\mathrm{span}(S) = \{j : j \in \mathrm{span}\{v_i : i \in S\}\}$$

usual lin alg.

$\mathbb{R}^2$



$\text{span}(\{1\}) = \{1,2\}$.

$\uparrow$ matroid span.

$\text{span}(\{v_1\}) = \{\alpha v_1 : \alpha \in \mathbb{R}\} \cong \mathbb{R}$

- **Claim:** $r(S) = r(\text{span}(S))$.

(rank is preserved by adding all elts that don't increase rank individually.)

Pf:

- Take $J \subseteq S$ max'l indep
- Suppose $r(\text{span}(S)) > |J|$

(P2) $\Rightarrow \exists\, e \in \text{span}(S) \setminus J$ s.t. $J + e \in \mathcal{I}$

$\Rightarrow r(S+e) \geq r(J+e) = |J|+1 > r(S)$

contradicts $e \in \text{span}(S)$. $\square$

- Say S is **closed** if $\text{span}(S) = S$;
  A.K.A S is a flat of M.

Matroid polytope

- Let $M = (E, \mathcal{I})$ matroid.
- Let $X = \{ \mathbb{1}_S : S \in \mathcal{I} \}$.
  $= \left\{ \begin{array}{l} \text{indicator vectors} \\ \text{of independent sets.} \end{array} \right\}$.
- the **matroid polytope** is

$$\boxed{P_M := \text{conv}(X)}$$

? inequalities of $P_M = \{ Ax \leq b, x \geq 0 \}$.

- some constraints: $\forall S \subseteq E$, $\mathbb{1}_{S'} \in X$

$$\mathbb{1}_{S'} \cdot \mathbb{1}_S = |S' \cap S| \leq r(S)$$

independent!

need constraints

$$\mathbb{1}_S \cdot x \leq r(S) \quad \forall S \subseteq E$$

Theorem: For r rank function of M, let

$$P = \{ x \in \mathbb{R}^E :$$

(rank) $\quad x(S) \leq r(S) \quad \forall S \subseteq E$

(nonnegativity) $\quad x_e \geq 0 \quad \forall e \in E \}.$

Here $x(S) = \sum_{e \in S} x_e = \mathbb{1}_S \cdot x$

$$\boxed{\text{Then } P_M = P.}$$

Notes:

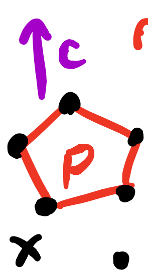
- We saw $P_M = \text{conv}(X) \subseteq P$
  b/c X satisfies all constraints
- Harder to show $P \subseteq P_M = \text{conv}(X)$
  ▷ use "3 techniques"

Algorithmic proof:

A.K.A primal-dual

- based on greedy alg.
- $\text{conv}(X) \subseteq P \Rightarrow$

$$\max\{c^T x : x \in X\} \leq \max\{c^T x : x \in P\}$$



- Enough to show this is equality.
  would follow if we find $x \in X$
  and dual feasible y s.t.

$$c^T x = b^T y.$$

weak duality

(because $c^T x \le \max\{c^T x : x \in P\} \overset{?}{\le} b^T y$

is equalities all the way across.).

NEXT TIME.

- What's the dual?
-

(primal) = (dual)

- Our primal:

$\max c^T x$

$$A = \begin{array}{cc} & e \\ s & \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} \end{array} 2^E$$

$(\qquad\qquad\qquad\qquad).$

- **Dual**: min

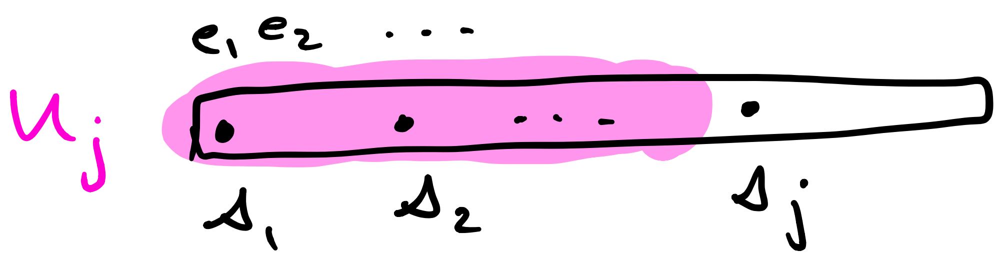
- Thus we need

- Consider cost c.
- max cost indep set =

- Need

- For $j \le k$,
- $u_j :=$

$=$



- Note

▷

▷

- For $j = 1 \dots k$, set

$$y_{U_j} :=$$

where

- Set

- **Claim 1:** y dual feasible.

Pf: $\triangleright$

$\triangleright$

- **Claim 2:** $\sum_{S \subseteq E} r(S) y_S = c(S_k)$.

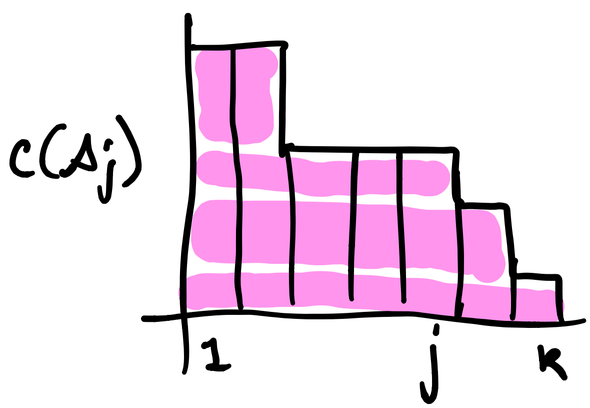
Pf: $\sum_{S \subseteq E} r(S) y_S =$

$=$

$=$

$=$ $\square$

Intuition: $c(S_k)$ is <u>area</u>



Vertex proof